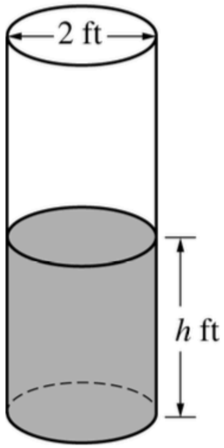
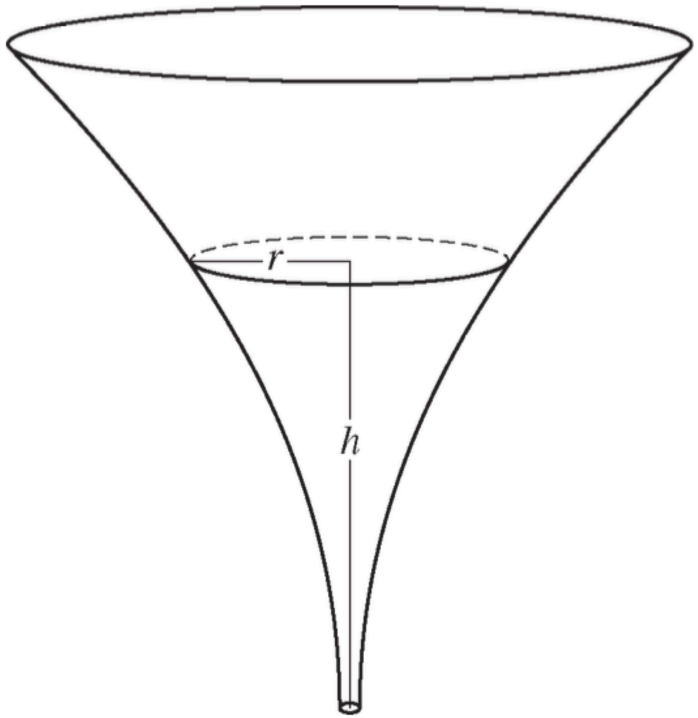


4. A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

   (a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

   (b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

   (c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t.



5. The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height h, the radius of the funnel is given by $r = \frac{1}{20}\left(3 + h^2\right)$, where $0 \le h \le 10$. The units of r and h are inches.

(a) Find the average value of the radius of the funnel.

(b) Find the volume of the funnel.

(c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters / minute)	0	100	40	–120	–150

4. Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

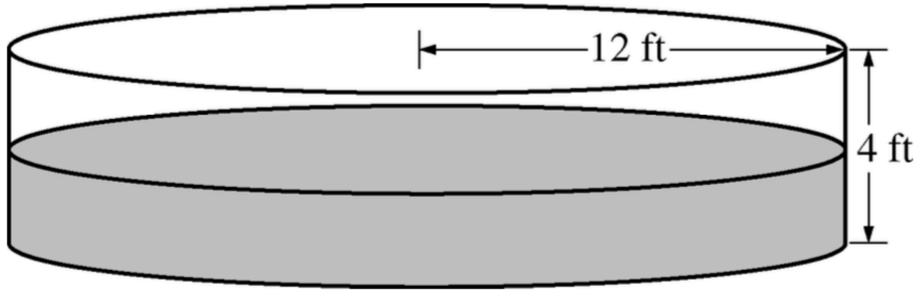
(a) Find the average acceleration of train A over the interval $2 \le t \le 8$.

(b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

(c) At time $t = 2$, train A's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

(d) A second train, train B, travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.

t	0	2	4	6	8	10	12
$P(t)$	0	46	53	57	60	62	63



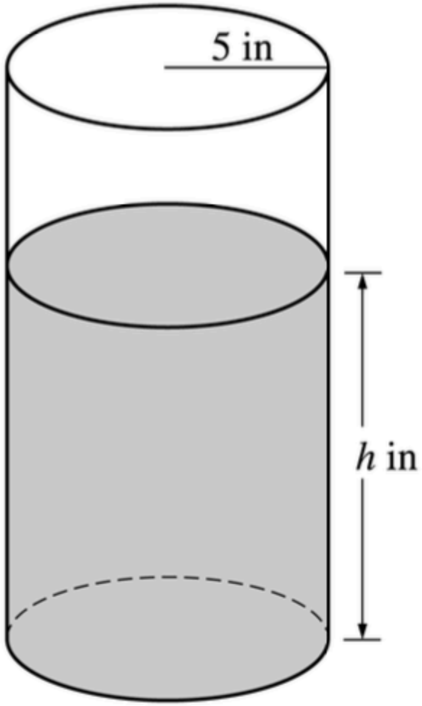
3. The figure above shows an aboveground swimming pool in the shape of a cylinder with a radius of 12 feet and a height of 4 feet. The pool contains 1000 cubic feet of water at time $t = 0$. During the time interval $0 \le t \le 12$ hours, water is pumped into the pool at the rate $P(t)$ cubic feet per hour. The table above gives values of $P(t)$ for selected values of t. During the same time interval, water is leaking from the pool at the rate $R(t)$ cubic feet per hour, where $R(t) = 25e^{-0.05t}$. (Note: The volume V of a cylinder with radius r and height h is given by $V = \pi r^2 h$.)

   (a) Use a midpoint Riemann sum with three subintervals of equal length to approximate the total amount of water that was pumped into the pool during the time interval $0 \le t \le 12$ hours. Show the computations that lead to your answer.

   (b) Calculate the total amount of water that leaked out of the pool during the time interval $0 \le t \le 12$ hours.

   (c) Use the results from parts (a) and (b) to approximate the volume of water in the pool at time $t = 12$ hours. Round your answer to the nearest cubic foot.

   (d) Find the rate at which the volume of water in the pool is increasing at time $t = 8$ hours. How fast is the water level in the pool rising at $t = 8$ hours? Indicate units of measure in both answers.



5. A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure above. Let h be the depth of the coffee in the pot, measured in inches, where h is a function of time t, measured in seconds. The volume V of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

(a) Show that $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$.

(b) Given that $h = 17$ at time $t = 0$, solve the differential equation $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$ for h as a function of t.

(c) At what time t is the coffeepot empty?

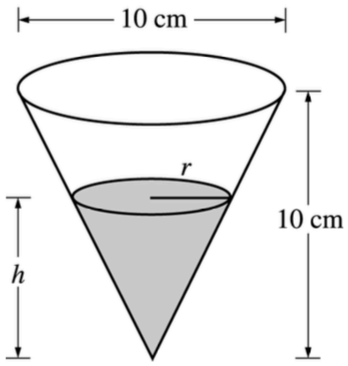
5. Consider the curve given by $y^2 = 2 + xy$.

   (a) Show that $\dfrac{dy}{dx} = \dfrac{y}{2y - x}$.

   (b) Find all points (x, y) on the curve where the line tangent to the curve has slope $\dfrac{1}{2}$.

   (c) Show that there are no points (x, y) on the curve where the line tangent to the curve is horizontal.

   (d) Let x and y be functions of time t that are related by the equation $y^2 = 2 + xy$. At time $t = 5$, the value of y is 3 and $\dfrac{dy}{dt} = 6$. Find the value of $\dfrac{dx}{dt}$ at time $t = 5$.



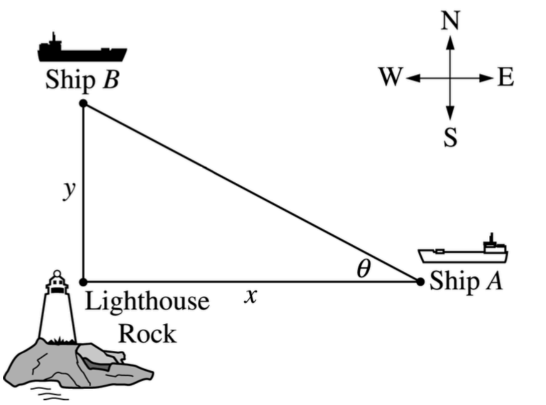
5. A container has the shape of an open right circular cone, as shown in the figure above. The height of the container is 10 cm and the diameter of the opening is 10 cm. Water in the container is evaporating so that its depth h is changing at the constant rate of $\frac{-3}{10}$ cm/hr.

(Note: The volume of a cone of height h and radius r is given by $V = \frac{1}{3}\pi r^2 h$.)

(a) Find the volume V of water in the container when $h = 5$ cm. Indicate units of measure.

(b) Find the rate of change of the volume of water in the container, with respect to time, when $h = 5$ cm. Indicate units of measure.

(c) Show that the rate of change of the volume of water in the container due to evaporation is directly proportional to the exposed surface area of the water. What is the constant of proportionality?



6. Ship A is traveling due west toward Lighthouse Rock at a speed of 15 kilometers per hour (km/hr). Ship B is traveling due north away from Lighthouse Rock at a speed of 10 km/hr. Let x be the distance between Ship A and Lighthouse Rock at time t, and let y be the distance between Ship B and Lighthouse Rock at time t, as shown in the figure above.

   (a) Find the distance, in kilometers, between Ship A and Ship B when $x = 4$ km and $y = 3$ km.

   (b) Find the rate of change, in km/hr, of the distance between the two ships when $x = 4$ km and $y = 3$ km.

   (c) Let θ be the angle shown in the figure. Find the rate of change of θ, in radians per hour, when $x = 4$ km and $y = 3$ km.

3. Oil is leaking from a pipeline on the surface of a lake and forms an oil slick whose volume increases at a constant rate of 2000 cubic centimeters per minute. The oil slick takes the form of a right circular cylinder with both its radius and height changing with time. (Note: The volume V of a right circular cylinder with radius r and height h is given by $V = \pi r^2 h$.)

   (a) At the instant when the radius of the oil slick is 100 centimeters and the height is 0.5 centimeter, the radius is increasing at the rate of 2.5 centimeters per minute. At this instant, what is the rate of change of the height of the oil slick with respect to time, in centimeters per minute?

   (b) A recovery device arrives on the scene and begins removing oil. The rate at which oil is removed is $R(t) = 400\sqrt{t}$ cubic centimeters per minute, where t is the time in minutes since the device began working. Oil continues to leak at the rate of 2000 cubic centimeters per minute. Find the time t when the oil slick reaches its maximum volume. Justify your answer.

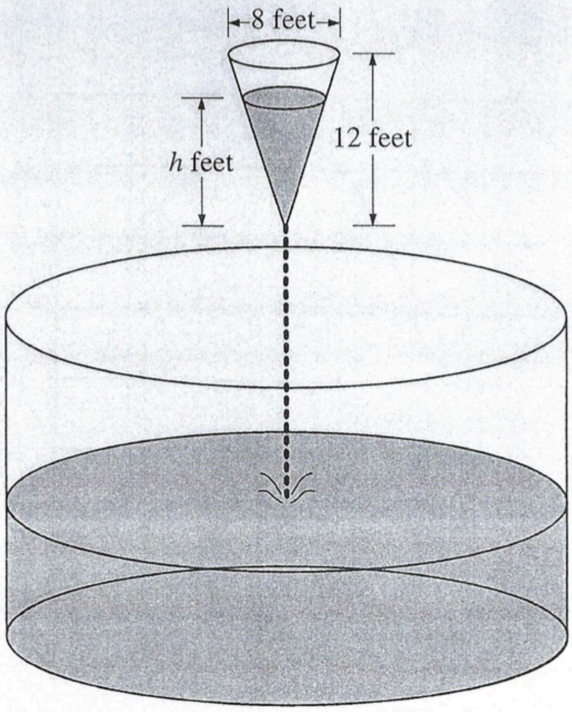
   (c) By the time the recovery device began removing oil, 60,000 cubic centimeters of oil had already leaked. Write, but do not evaluate, an expression involving an integral that gives the volume of oil at the time found in part (b).

4. The radius r of a sphere is increasing at a constant rate of 0.04 centimeters per second.
$\left(\underline{\text{Note:}}\ \text{The volume of a sphere with radius } r \text{ is } V = \frac{4}{3}\pi r^3.\right)$

(a) At the time when the radius of the sphere is 10 centimeters, what is the rate of increase of its volume?

(b) At the time when the volume of the sphere is 36π cubic centimeters, what is the rate of increase of the area of a cross section through the center of the sphere?

(c) At the time when the volume and the radius of the sphere are increasing at the same numerical rate, what is the radius?

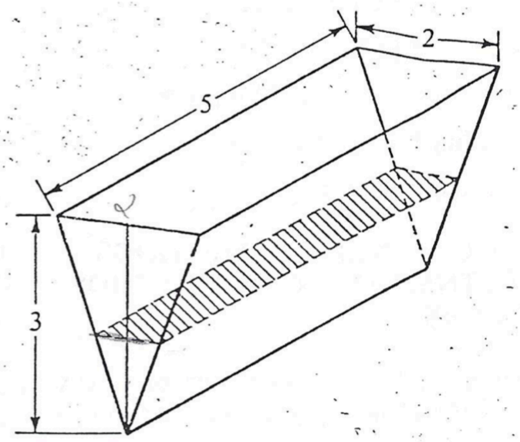


5. As shown in the figure above, water is draining from a conical tank with height 12 feet and diameter 8 feet into a cylindrical tank that has a base with area 400π square feet. The depth h, in feet, of the water in the conical tank is changing at the rate of $(h - 12)$ feet per minute. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

(a) Write an expression for the volume of water in the conical tank as a function of h.

(b) At what rate is the volume of water in the conical tank changing when $h = 3$? Indicate units of measure.

(c) Let y be the depth, in feet, of the water in the cylindrical tank. At what rate is y changing when $h = 3$? Indicate units of measure.



5. The trough shown in the figure above is 5 feet long, and its vertical cross sections are inverted isosceles triangles with base 2 feet and height 3 feet. Water is being siphoned out of the trough at the rate of 2 cubic feet per minute. At any time t, let h be the depth and V be the volume of water in the trough.

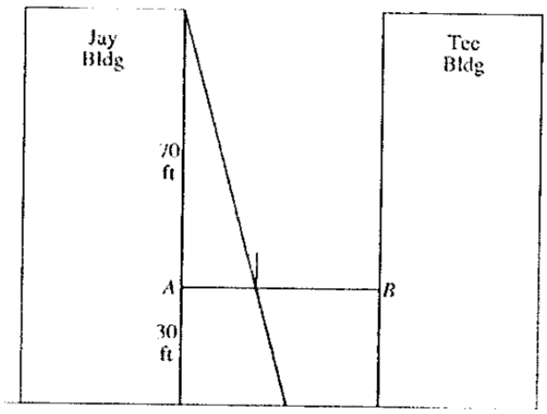
(a) Find the volume of water in the trough when it is full.

(b) What is the rate of change in h at the instant when the trough is $\frac{1}{4}$ full by volume?

(c) What is the rate of change in the area of the surface of the water (shaded in the figure) at the instant when the trough is $\frac{1}{4}$ full by volume?

1991 AB6

A tight rope is stretched 30 feet above the ground between the Jay and the Tee buildings, which are 50 feet apart. A tightrope walker, walking at a constant rate of 2 feet per second from point A to point B, is illuminated by a spotlight 70 feet above point A, as shown in the diagram.



(a) How fast is the shadow of the tightrope walker's feet moving along the ground when she is midway between the buildings? (Indicate units of measure.)

(b) How far from point A is the tightrope walker when the shadow of her feet reaches the base of the Tee Building? (Indicate units of measure.)

(c) How fast is the shadow of the tightrope walker's feet moving up the wall of the Tee Building when she is 10 feet from point B? (Indicate units of measure.)

3.8 Maximize or Minimize?

743. The famous Kate Lynn Horsefeed is building a box as part of her science project. It is to be built from a rectangular piece of cardboard measuring 25 cm by 40 cm bycutting out a square from each corner and then bending up the sides. Find the size of the corner square which will produce a container that will hold the most.

744. Ashley is building a window in the shape of an equilateral triangle whose sides each measure 4 meters. Ashley wants to inscribe a rectangular piece of stained glass in the triangle, so that two of the vertices of the rectangle lie on one of the sides of the triangle. Find the dimensions of the rectangle of maximum area that can be inscribed in the given triangle.

745. It has been determined by the brilliant deductive mind of Bruce Wayne that Gotham Highway is located on the line $y = 2x + 3$. Determine the point on Gotham Highway closest to the Wayne Foundation Building, which happens to be located at the point $(1, 2)$.

746. Vaidehi wants to cut a 30-meter piece of iron into two pieces. One of the pieces will be used to build an equilateral triangle, and the other to build a rectangle whose length is three times its width. Where should Vaidehi cut the iron bar if the combined area of the triangle and the rectangle is to be a minimum? How could the combined area of these two figures be a maximum? Justify your answers.

747. An open oak wood box with a square base is to be constructed using 192 cm^2 of oak. If the volume of the box is to be maximized, find its dimensions.

748. At the Skywalker moisture farm on the desert planet Tatooine, there are 24 moisture processors, with an average yield per processor of 300 cubits of moisture. Research conducted at Mos Eisley University concludes that when an additional processor is used, the average yield per processor is reduced by 5 cubits. Help Owen and Beru Skywalker find the number of moisture processors that will maximize the number of cubits.

749. The fence around Wayne Manor is going to be replaced. No fence will be required on the side lying along Gotham River. If the new wrought iron fence costs \$12 per meter for the side parallel to the river, and \$4 per meter for the other two sides, find the dimensions of the maximum area that can be enclosed by the fence if Bruce Wayne cannot spend more than \$3600.

750. The Gotham-Metropolis Highway is a toll road that has averaged 54,000 cars per day over the past five years, with a \$.50 charge per car. A study conducted by the Ray Chulldel Lavet University concludes that for every \$.05 increase in the toll, the number of cars will be reduced by 500. In order to maximize revenue, what toll should the highway charge?

751. The range R of a projectile whose muzzle velocity in meters per second is v, and whose angle of elevation in radians is θ, is given by $R = (v^2 \sin(2\theta))/g$ where g is the acceleration of gravity. Which angle of elevation gives the maximum range of the projectile?

752. A piece of wire 100 cm long is to be cut into several pieces and used to construct the skeleton of a box with a square base.

a) What is the largest possible volume that such a box can have?

b) What is the largest possible surface area?

753. In medicine, the reaction $R(x)$ to a dose x of a drug is given by $R(x) = Ax^2(B-x)$, where $A > 0$ and $B > 0$. The sensitivity $S(x)$ of the body to a dose of size x is defined to be $R'(x)$. Assume that a negative reaction is a bad thing.

a) What seems to be the domain of R? What seems to be the physical meaning of the constant B? What seems to be the physical meaning of the constant A?

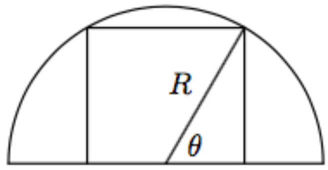
b) For what value of x is R a maximum?

c) What is the maximum value of R?

d) For what value of x is the sensitivity a minimum?

e) Why is it called sensitivity?

754. What is the area of the largest rectangle that can be inscribed in a semicircle of radius R so that one of the sides of the rectangle lies on the diameter of the semicircle?



755. An electronics store needs to order a total of 2400 CD players over the course of a year. It will receive them in several shipments, each containing an equal number of CD players. The shipping costs are \$50 for each shipment, plus a yearly fee of \$2 for each CD player in a single shipment. What size should each shipment be in order to minimize yearly shipping costs?

756. A rectangle in the first quadrant has one side on the y-axis, another on the x-axis, and its upper right-hand vertex on the curve $y = e^{-x^2}$. What is the maximum area of the rectangle?

757. The positions of two particles on the x-axis are $x_1 = \sin t$ and $x_2 = \sin(t + \frac{\pi}{3})$.

a) At what time(s) in the interval $[0, 2\pi]$ do the particles meet?

b) What is the farthest apart that the particles ever get?

c) When in the interval $[0, 2\pi]$ is the distance between the particles changing the fastest?

758. One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2}$$

where q is the quantity ordered when things run low, k is the cost of placing an order (a constant), m is the number of items sold each week (a constant), h is the weekly holding cost per item (a constant), and c is a constant. What is the quantity that will minimize $A(q)$? (The expression you get for your answer is called the *Wilson lot size formula.*)

759. The function $f(x) = \cot x - \sqrt{2}\csc x$ has an absolute maximum value on the interval $[0, \pi]$. Find its exact value.

When introduced at the wrong time or place, good logic may be the worst enemy of good teaching. —*George Polya*

726. (a) $\{x|x \neq \pm 3\}$ (b) 0 (c) 0 (d) min at $(0,0)$ (e) inc for $x < -3$ and $-3 < x < 0$, dec for $0 < x < 3$ and $x > 3$ (f) none (g) ccup for $x < -3$ and $x > 3$, ccdown for $-3 < x < 3$

729. (a) $\{x|x > 0\}$ (b) 1 (c) none (d) max at $(e, \frac{1}{e})$ (e) inc for $0 < x < e$, dec for $x > e$ (f) $(e^{3/2}, \frac{3}{2e^{3/2}})$ (g) ccup for $x > e^{3/2}$, ccdown for $0 < x < e^{3/2}$

732. 3.84

733. (a) mins at $x = -2.5$ and $x = 2$, max at $x = 0$ (b) ccup for $-3 < x < -1$ and $1 < x < 3$, ccdown for $-1 < x < 1$ and $3 < x < 4$

735. (a) at $t = \frac{2}{3}$ $x = \frac{53}{81}$, at $t = -1$ $x = \frac{1}{12}$ (b) at $t = \frac{1}{12}$ $x = -\frac{2}{3}\sqrt{3}$

736. (a) 0 (b) 6 (c) always right

737. (a) $v(t) = -2\pi t \sin(\frac{\pi}{2}t^2)$ (b) $a(t) = -2\pi(\sin(\frac{\pi}{2}t^2) + \pi t^2 \cos(\frac{\pi}{2}t^2))$ (c) right for $-1 < t < 0$, left for $0 < t < 1$ (d) 0

738. (a) $3\pi t - 3\pi t \cos(\frac{3\pi}{2}t^2)$ (b) $3\pi - 2\pi \cos(\frac{3\pi}{2}t^2) + 9\pi^2 t^2 \sin(\frac{3\pi}{2}t^2)$ (c) $0, \sqrt{\frac{4}{3}}, \sqrt{\frac{8}{3}}$ (d) $0, 2\pi, 4\pi$

740. (a) $4e^{3t} - 8$ (b) $12e^{3t}$ (c) $\frac{1}{3}\ln 2$ (d) $\frac{8}{3}(1 - \ln 2)$

741. (a) 135 sec (b) $\frac{5}{73}$ furlongs (c) $\frac{1}{13}$ furl/sec (d) the last and first furlong

746. one piece 14.8 m, other 15.2 m; use all iron to make the triangle

747. $8 \times 8 \times 4$ cm

748. 42

749. 225×150 m

750. \$2.95

751. $\frac{\pi}{4}$

752. (a) ≈ 578.7 cm^3 (b) $616\frac{2}{3}$ cm^2

753. (a) $[0, B]$, max dosage, scale factor (b) $\frac{2}{3}B$ (c) $\frac{4}{27}AB^3$ (d) $\frac{1}{3}B$

754. R^2

755. 10 shipments of 240 players each

756. $\frac{1}{\sqrt{2e}}$

757. (a) $\frac{\pi}{3}, \frac{4\pi}{3}$ (b) 1 (c) $\frac{\pi}{3}, \frac{4\pi}{3}$

759. -1

769. crit pt is $x = 1$, inc for $x < 1$, dec for $x > 1$, extrema at $x = 1$

778. (c) $\arctan x + \dfrac{x}{1 + x^2}$ (e) $-25x^{-2} + 6x^{-1/2}$ (f) $30x^4 - 60x^3 + 20x - 21$ (g) $\dfrac{-2(x^2+1)}{(x^2-1)^2}$

780. $y = \frac{1}{2}t$

781. (d) $y = e^x$

782. $y' = \cot x$

783. e^3

788. $\frac{\pi}{3}$

789. (a) 4, 0 (b) $-1, -1, 1, \frac{1}{2}$ (c) $0, -\frac{3}{2}$

790. (a) 1, $\frac{3}{4}$ (b) positive (c) zero

791. (a) $h' = 0$ (b) $k' = 0$

793. $\frac{1}{4}$

794. (a) odd (b) $\dfrac{1 + \cos x + x \sin x}{\cos^2 x}$ (c) $y = 2x$

795. A

796. E

798. (a) max at $x = -1$, mins at $x = \pm 3$ (b) $x = 0$, $x = 1$

799. (b) $x = 0$ (c) everywhere

800. $\mathbb{R}$, min at $(0, \frac{1}{10})$

801. $\mathbb{R}$, max at $(0, 10)$

803. $\{x|x \neq -1\}$, no extrema

804. $\{x|x > 0\}$, no extrema

806. $e^{-x}(x - 2)$

807. $e^x(x^2 + 4x + 2)$

808. $e^{x+e^x}(1 + e^x)$

810. (a) $\dfrac{-2xy}{x^2 + y^2}$ (b) $y = \frac{4}{5}x - \frac{13}{5}$ (c) $\sqrt[3]{-13}$

811. (a) $0, \frac{\pi}{2}, \pi$ (b) $\frac{\pi}{6} < x < \frac{\pi}{2}$ and $\frac{5\pi}{6} < x < \frac{3\pi}{2}$ (c) min of $-\frac{1}{4}$, max of 2

812. (a) $y = 4x + 2$ and $y = 4x - 2$ (b) 1 (c) 0

813. (a) $x = -2$ (b) $x = 4$ (c) $-1 < x < 1$ and $3 < x < 5$

814. (a) $\{x|x \neq 0\}$ (b) even (c) maxs at $x = \pm 1$ (d) $f(x) \leq \ln \frac{1}{2}$

815. (b) $c \approx 1.579$ (c) $y \approx 1.457x - 1.075$ (d) $y \approx 1.457x - 1.579$

817. (a) $k = -2$, $p = 2$ (b) always inc (c) $(1, 1)$

818. (a) min of $\dfrac{-e^{5\pi/4}}{\sqrt{2}}$, max of $e^{2\pi}$ (b) inc for $0 < x < \frac{\pi}{4}$ and $\frac{5\pi}{4} < x < 2\pi$ (c) π

819. (a) 100 (b) $y = \frac{3}{5}x + 20$ (c) yes, the top 5 ft of the tree

820. C

821. B

822. B

823. A

824. D

825. C

826. D

827. C

828. D

829. E

830. B

831. E